**SOLUTIONS**



# The University of Nottingham

DEPARTMENT OF MECHANICAL, MATERIALS, AND MANUFACTURING ENGINEERING

A LEVEL 2 MODULE, AUTUMN SEMESTER 2012-2013

**MECHANICS OF SOLIDS 2**

Time allowed ONE Hour and THIRTY Minutes

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*Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced*

***Answer THREE questions***

*Only silent, self contained calculators with a Single-Line Display or Dual-Line Display are permitted in the examination.*

*Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.*

*No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.*

***DO NOT turn examination paper over until instructed to do so***

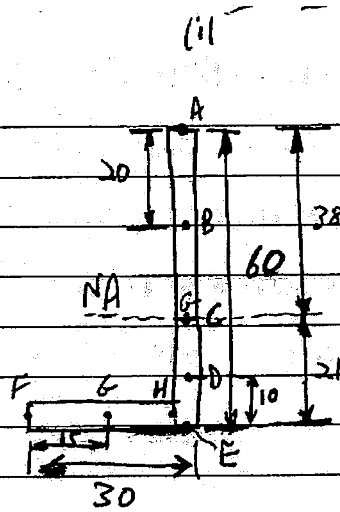
**ADDITIONAL MATERIAL:** Graph Paper

**INFORMATION FOR INVIGILATORS:**

**SOLUTIONS**

(i)

MM2MS2 Aut 1213




$I = 156{,}940 \text{ mm}^4$

$S = 1000 \text{ N}$

Flange – horizontal shear stresses

F – free surface – $\tau = 0$

At distance a from F $\quad \tau = \dfrac{S . a\cancel{t} . (21.9 - 2.5)}{I \cancel{t}}$

$= \dfrac{1000\,(21.9 - 2.5)\,a}{156{,}940}$

$= 0.124\,a$

At G $\quad \tau = 0.124 . 15 = 1.86 \text{ MPa}$

At H $\quad \tau = 0.124 . 25 = 3.1 \text{ MPa}$ [13 Marks]

Web – vertical shear stresses

A & E – free surfaces – $\tau = 0$

At C [calculated using top half] $\quad \tau = \dfrac{1000 \times 38.1 \times \cancel{5} \times 38.1/2}{156{,}940 \times \cancel{5}} = 4.62 \text{ MPa}$

At B $\quad \tau = \dfrac{1000 \times 20 \times \cancel{5} \times 38.1/40}{156{,}940 \times \cancel{5}} = 3.70 \text{ MPa}$

At D $\quad \tau = \dfrac{1000 \times [10 \times 5 \times 16.9 + 25 \times 5 \times 19.4]}{156{,}940 \times 5} = 4.17 \text{ MPa}$

[13 marks]

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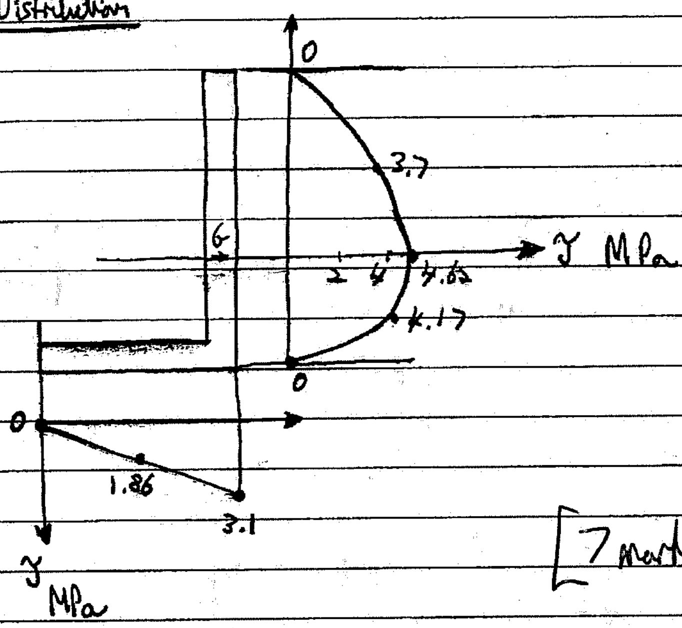
(ii)

MM2MS2 Aut 1213

Sketch of Distribution




[7 marks]

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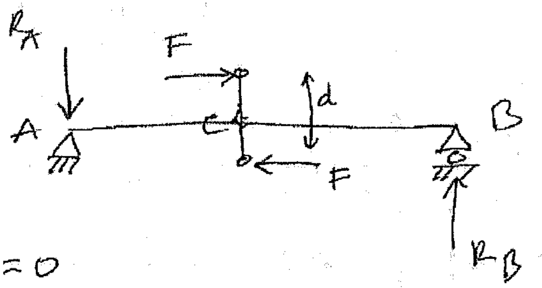
(2)

(a)

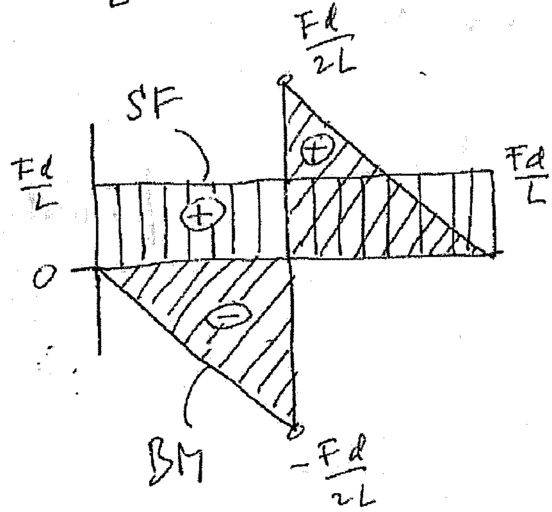



$\Sigma M = 0$

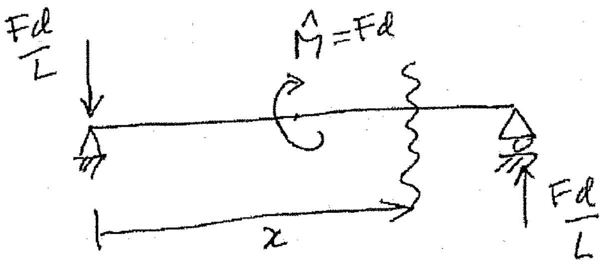
$R_A = \frac{Fd}{L}$ (↓)

$R_B = \frac{Fd}{L}$ (↑)

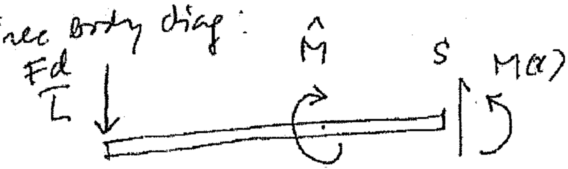
[8 marks]




(b)




Free body diag:




$M(x) = \hat{M}\langle x - \frac{L}{2}\rangle^0 - \frac{Fd}{L}x$

Beam eq.:

$$EI\frac{d^2y}{dx^2} = -M$$
$$= \frac{Fd}{L}x - \hat{M}\langle x - \tfrac{L}{2}\rangle^0$$

$$EI\frac{dy}{dx} = \frac{Fd}{2L}x^2 - Fd\langle x - \tfrac{L}{2}\rangle + C_1$$

$$EI\,y = \frac{Fd}{6L}x^3 - \frac{Fd}{2}\langle x - \tfrac{L}{2}\rangle^2 + C_1x + C_2$$

**SOLUTIONS**

BCs:

at $x=0 \rightarrow y=0 \Rightarrow C_2=0$

at $x=L \rightarrow y=0 \Rightarrow \frac{Fd}{6L}L^3 - \frac{Fd}{2}\frac{L^2}{4} + C_1 L = 0$

$$\therefore C_1 = -\frac{FdL}{24}$$

Beam deflection:

$$EI\,y = \frac{Fd}{6L}x^3 - \frac{Fd}{2}\langle x-\tfrac{L}{2}\rangle^2 - \frac{FdL}{24}x$$

at C ($x=\frac{L}{2}$):

$$y = \underline{\underline{0}}$$

Beam rotation:

$$EI\frac{dy}{dx} = \frac{Fd}{2L}x^2 - Fd\langle x-\tfrac{L}{2}\rangle - \frac{FdL}{24}$$

[13 marks]

at C ($x=\frac{L}{2}$):

$$\frac{dy}{dx} = \frac{FdL}{12EI} \quad \text{cw}$$

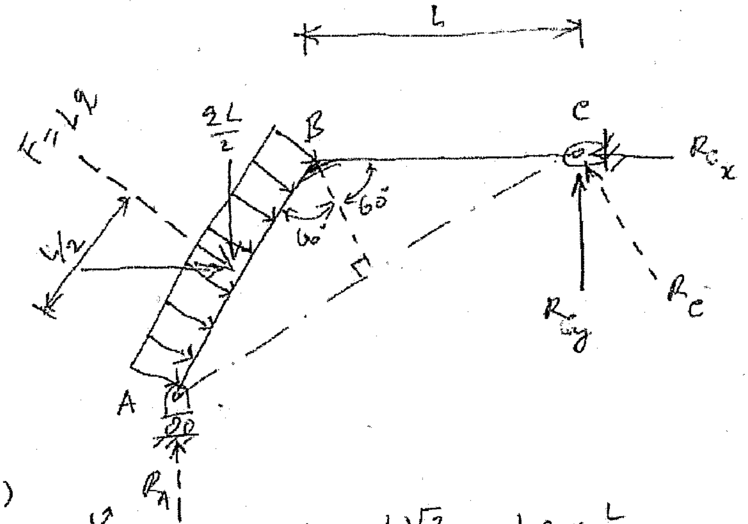
(c) $$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{FdL}{24EI} \quad \text{ccw}$$

$$\left.\frac{dy}{dx}\right|_{x=L} = -\frac{FdL}{24EI} \quad \text{ccw}$$


[12 marks]

**SOLUTIONS**

① 

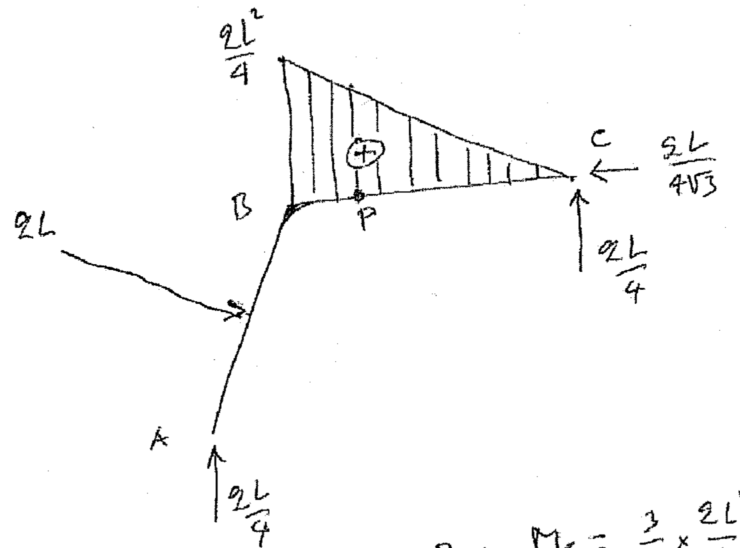



(a) $\sum \overset{\circlearrowleft}{M}_A = 0$ : $R_C \times L\sqrt{3} = Lq \times \frac{L}{2}$

$\therefore R_C = \frac{qL}{2\sqrt{3}} \rightarrow R_{Cx} = \frac{qL}{4\sqrt{3}}$ (←) & $R_{Cy} = \frac{qL}{4}$ (↑)

$\sum \vec{F}|_y = 0$ : $R_A = \frac{qL}{2} - R_{Cy} = \frac{qL}{4}$ (↑) [10 marks]

(b)




[10 marks]

(c) Bending moment at P : $M_P = \frac{3}{4} \times \frac{qL^2}{4} = \frac{3}{16} qL^2$

Bending stress at P = $\sigma_B = \dfrac{\frac{3}{16} qL^2 \times \frac{d}{2}}{\frac{1}{12} d h^3} = \frac{9}{16} \frac{qL^2}{h^3}$

Normal stress at P = $\sigma_n = \dfrac{\frac{qL}{4\sqrt{3}}}{dh} = \frac{\sqrt{3}}{12} \frac{qL}{dh}$

$\therefore$ Total compressive stress at P = $\sigma_B + \sigma_n$

$= \frac{9}{16} \frac{qL^2}{h^3} + \frac{\sqrt{3}}{12} \frac{qL}{dh}$

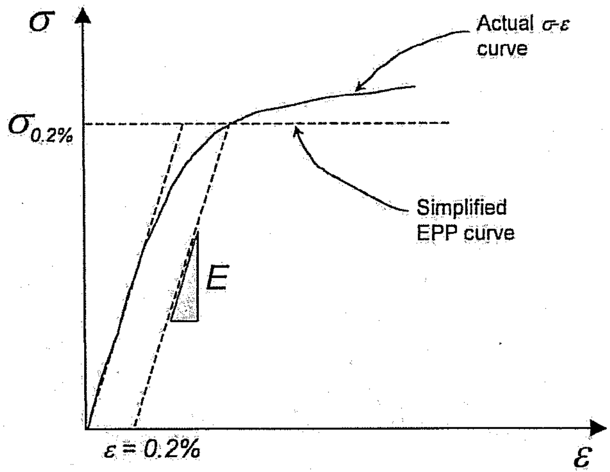
[13 marks]

1(a)

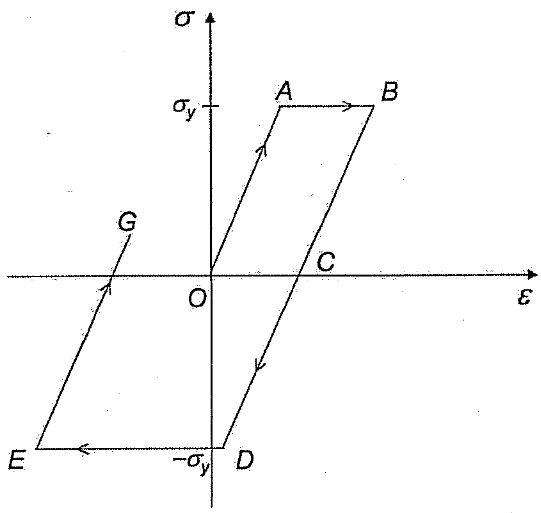
**Solution**

Elastic-perfectly-plastic (EPP)

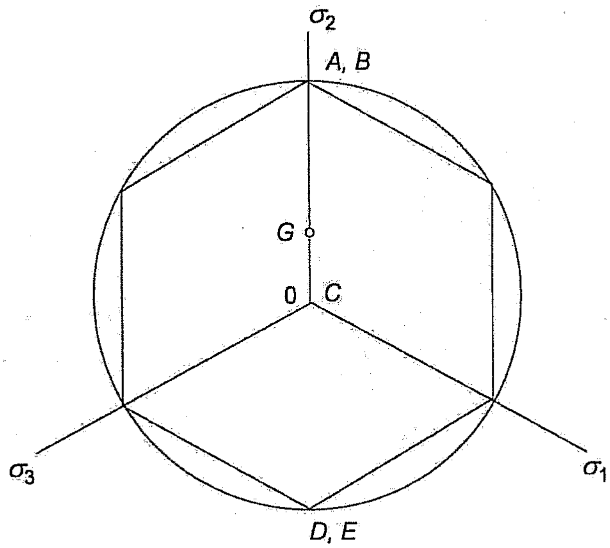
In this case there is no hardening, i.e. the yield stress, $\sigma_y$, remains constant at $\pm\sigma_y$, regardless of any previous plastic deformation. Therefore, the yield surface doesn't change in either shape or position in the principal stress-space.




E-P-P Stress-Strain Curve




Uniaxial cyclic σ-ε behaviour for an EPP material model



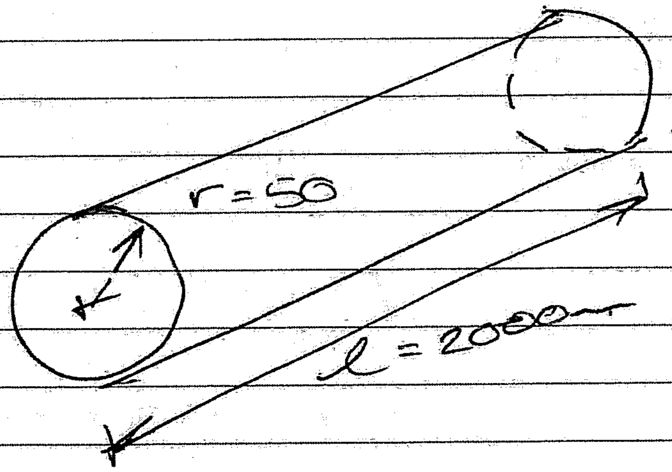

Representation of uniaxial stress-strain behaviour in principal stress space

[7 marks]

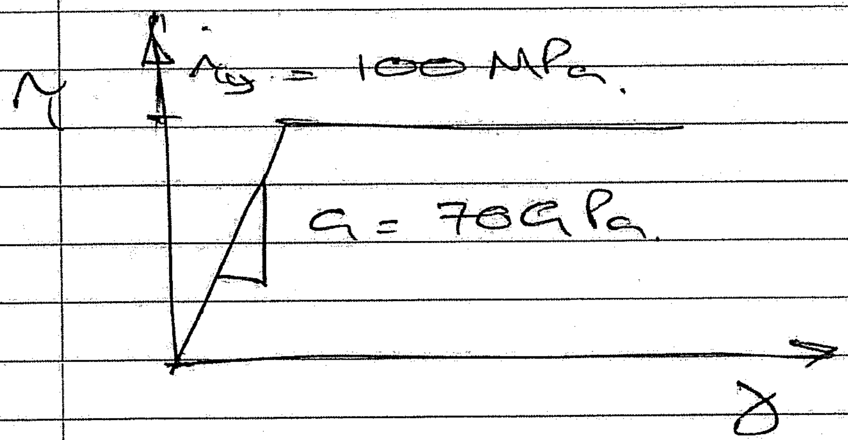


1

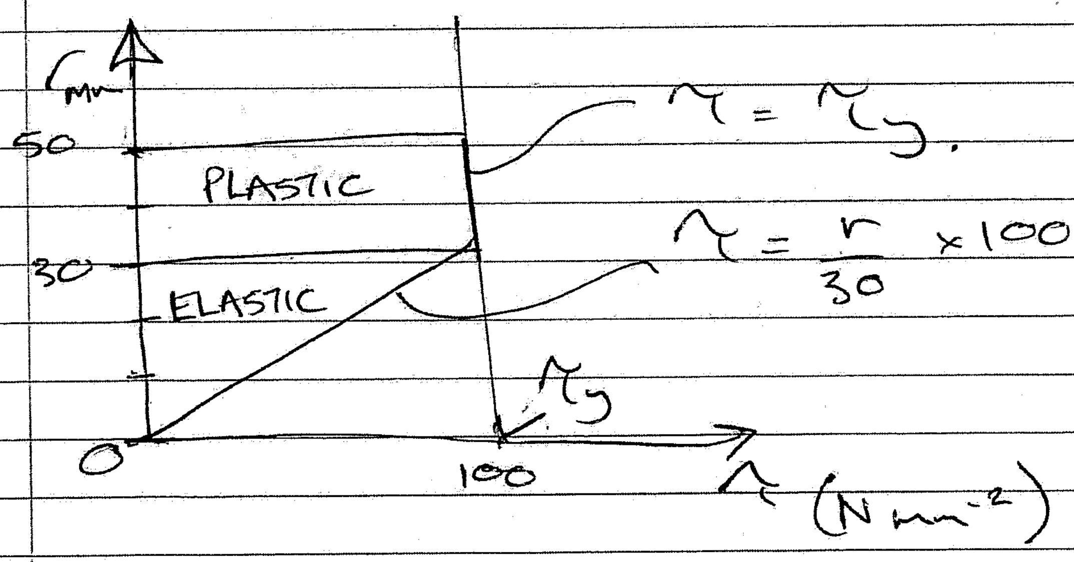
1(b)



r = 50
l = 2000mm

τ
τy = 100 MPa
G = 70GPa
γ

r mm
50
30
0
PLASTIC
ELASTIC
τ = τy
τ = r/30 × 100
τy
100
τ (N mm⁻²)

(1) <u>Equilibrium</u>

$$T = \int_{0}^{50} r \times \tau \times 2\pi r \, dr$$

$$= \int_{0}^{50} 2\pi \tau r^2 \, dr$$

$$= \int_{0}^{30} 2\pi \times \frac{r}{30} \times 100 \times r^2 dr. \quad \text{(elastic)}$$

$$+ \int_{30}^{50} 2\pi \times \tau_y r^2 dr \quad \text{(plastic)}$$

$$= \frac{20\pi}{3}\int_{0}^{30} r^3 dr + 200\pi \int_{30}^{50} r^2 dr.$$

$$= \frac{20\pi}{3}\left[\frac{r^4}{4}\right]_0^{30} - 200\pi \left[\frac{r^3}{3}\right]_{30}^{50}$$

$$= \frac{20\pi}{12} \times 30^4 + \frac{200\pi}{3}\left[50^3 - 30^3\right]$$

$$= 4.241 \times 10^6 + 20.525 \times 10^6 \text{ MPa}.$$

$$T = 24.76 \text{ kNm}$$

(3).

## Compatability

At $r = 30\text{ mm}$ $\tau = \tau_y$, $\gamma = \gamma_y$.

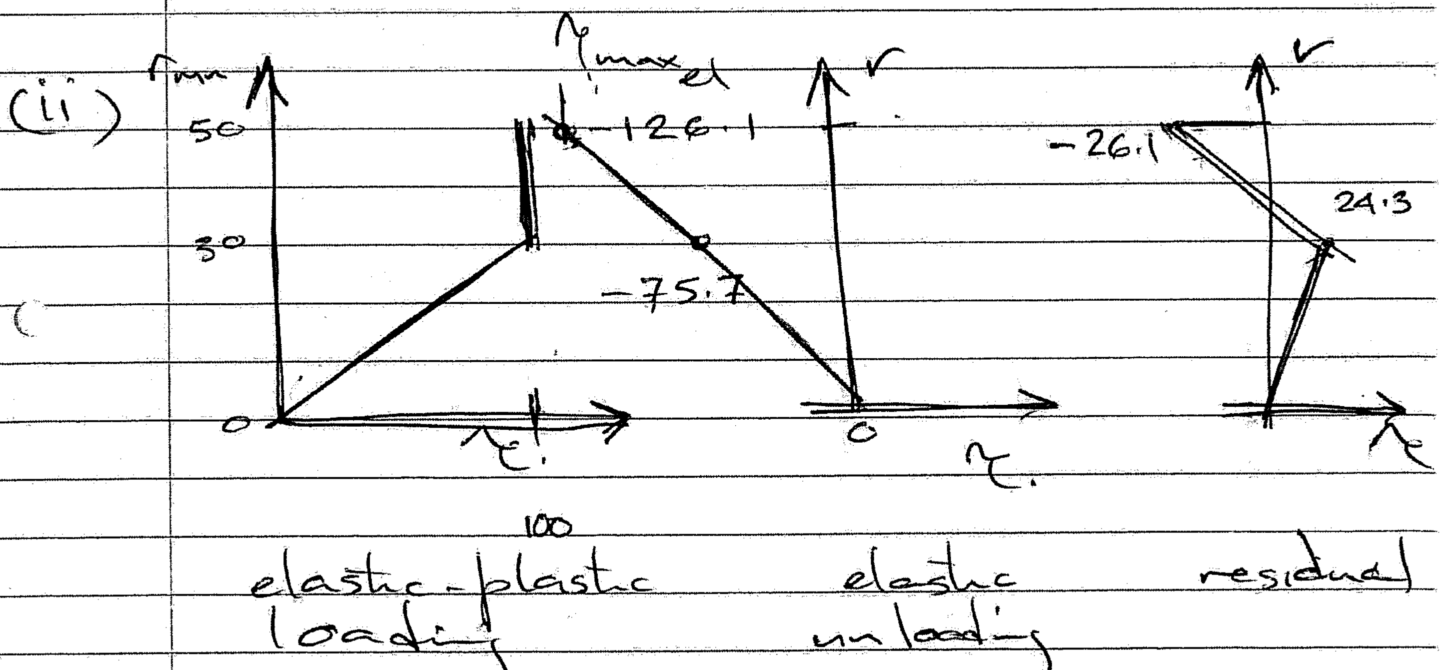
and $G = \dfrac{\tau_y}{\gamma_y}$ ie $\gamma_y = \dfrac{\tau_y}{G}$.

$$\therefore \quad \gamma_y = \frac{100\text{ N mm}^{-2}}{70 \times 10^3\text{ N mm}^{-2}} = 1.4286 \times 10^{-3}$$

$$\therefore \quad \theta = \frac{\gamma l}{r} = 1.42 \times 10^{-3} \times \frac{2000}{30}.$$ [13 marks]

$$\theta = 0.09524\text{ rad} = \underline{5.457^\circ}$$

(ii)




Elastic unloading $\tau^{max}_{el} = \dfrac{Tr}{J}$.

$$J = \frac{\pi d^4}{32} = \frac{\pi \times 100^4}{32} = 9.817 \times 10^6\text{ mm}^4$$

$$\tau_{el}^{max} = \frac{-24.76 \times 10^6 \ Nmm \times 50}{9.817 \times 10^6 \ mm^4}$$

$$\tau_{el}^{max} = \underline{-126.13 \ Nmm^{-2}}$$

$$\therefore (\tau_{el})_{r=30} = \frac{30}{50} \times (-126.13) \ Nmm^{-2}$$

$$= \underline{-75.68 \ Nmm^{-2}}$$

Residual angle of twist determined at $r = 30mm$ (boundary of onset of plasticity)

$$\gamma_{r=30mm} = \frac{\tau_{residual}}{G} = \frac{24.32}{70 \times 10^3}$$

$$= \underline{3.474 \times 10^{-4}}$$

$$\theta_{residual} = \frac{3.474 \times 10^{-4} \times 2000}{30} \times \left(\frac{360}{2\pi}\right)$$

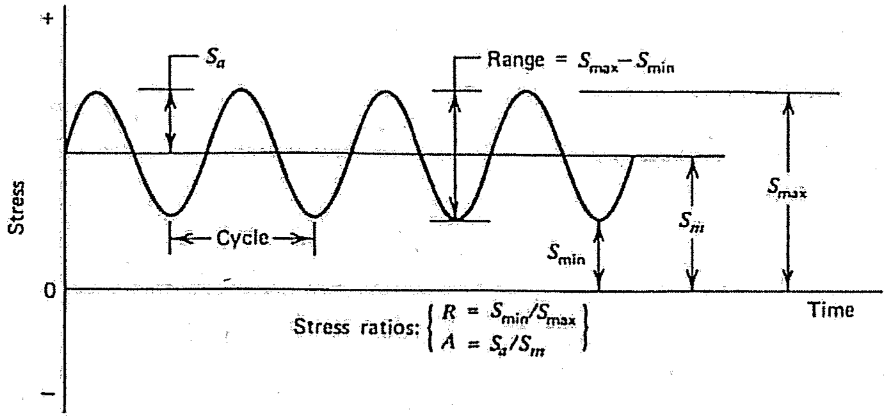
$$= \underline{1.327^\circ}$$

[13 marks]

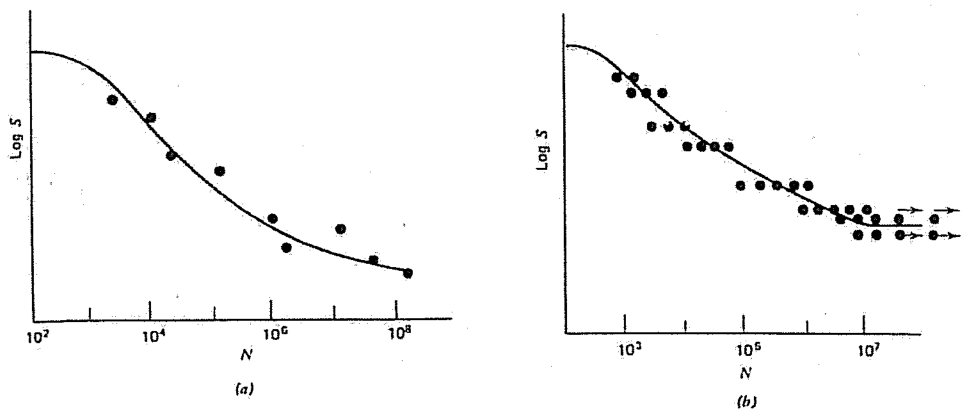
2(a)

**S-N Total life approach**

The total life approach is based on the results of stress- and strain-controlled cyclic testing of laboratory test specimens of material, in order to obtain the numbers of cycles to failure as a function of the applied alternating stress. Most fatigue testing was based on fully-reversed (i.e. zero mean stress, $S_m = 0$), stress-controlled conditions and the fatigue design data was presented in the form of *S-N* curves, which are either semi-log or log-log plots of alternating stress, $S_a$, against the measured number of cycles to failure, *N*, where failure is defined as fracture.

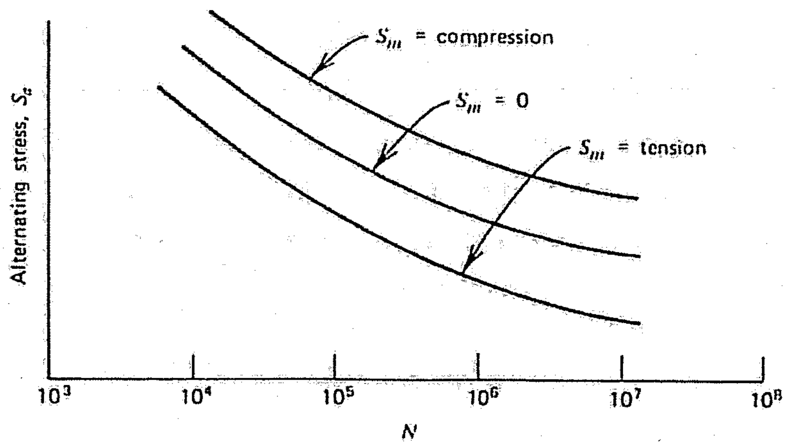



Schematic representations of two typical *S-N* curves obtained under load or stress-controlled tests on smooth specimens: a continuously sloping curve and a discontinuity or "knee" in the curve. A "knee" is only found in a few materials (notably low strength steels) between $10^6$ and $10^7$ cycles under non-corrosive conditions. The curves are normally drawn through the median life value (of the scatter in *N*) and thus represents 50 percent expected failure. The *fatigue life*, *N*, is the number of cycles of stress or strain range of a specified character that a given specimen sustains before failure of a specified nature occurs. *Fatigue strength* is a hypothetical value of stress range at failure for exactly *N* cycles as obtained from an *S-N* curve. The *fatigue limit* (sometimes called the *endurance limit*) is the limiting value of the median fatigue strength as *N* becomes very large, eg >$10^8$ cycles.

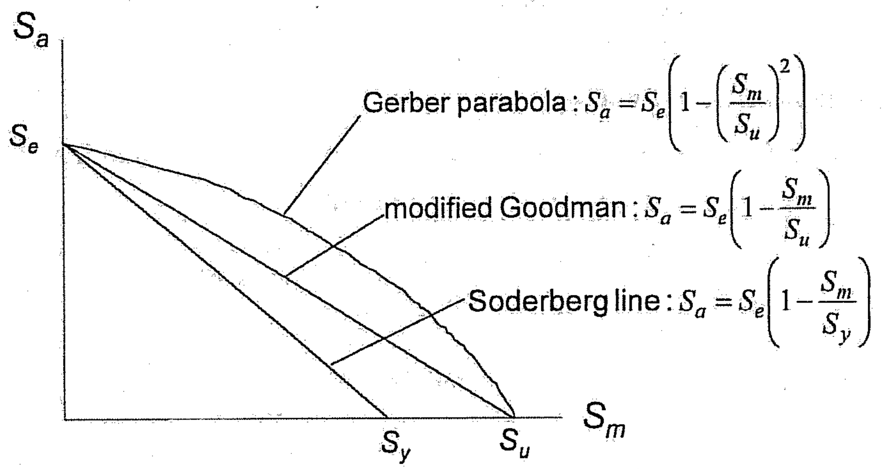



Typical S-N diagrams.

**Effect of mean stress** the mean stress has a significant effect on fatigue behaviour. It can be seen that tensile mean stresses are detrimental while compressive mean stresses are beneficial.




The effect of mean stress on fatigue life.




The effect of mean stress on fatigue life.

The effect of mean stress is commonly represented as a plot of $S_a$ versus $S_m$ for a given fatigue life. Attempts have been made to develop this relationship into general relations. Three of these common relations between allowable alternating stress for a given life as a function of mean stress, are shown in Figure 3.11.7. The modified Goodman line assumes a linear relationship between the allowable Sa and the corresponding mean stress $S_m$, where the slope and intercepts are defined by the fatigue life, $S_e$, and the material UTS, $S_u$, respectively. The Gerber parabola employs the same end-points but, in this case, the relation is defined by a parabola. Finally, the Soderberg line again assumes a linear relation, but this time the mean stress axis end-point is taken as the yield stress, $S_y$. The modified Goodman line, for example, can be extended into the compressive mean stress region to give increasing allowable alternating stress with increasing compressive mean stress, but this is normally taken to be horizontal for design purposes and for conservatism.

**Effect of stress concentrations** are expressed in terms of an elastic stress concentration factor (SCF), $K_t$, which is simply the relationship between the maximum local stress and an appropriate nominal stress, as follows:

$$K_t = \frac{\sigma_{max}^{el}}{\sigma_{nom}}$$

It was once thought that the fatigue strength of a notched component could be predicted as the strength of a smooth component divided by the SCF. The reduction is, in fact, often less than $K_t$ and is defined by the *fatigue notch factor*, $K_f$, which is defined as the ratio of the smooth fatigue strength to the notched fatigue strength as follows:

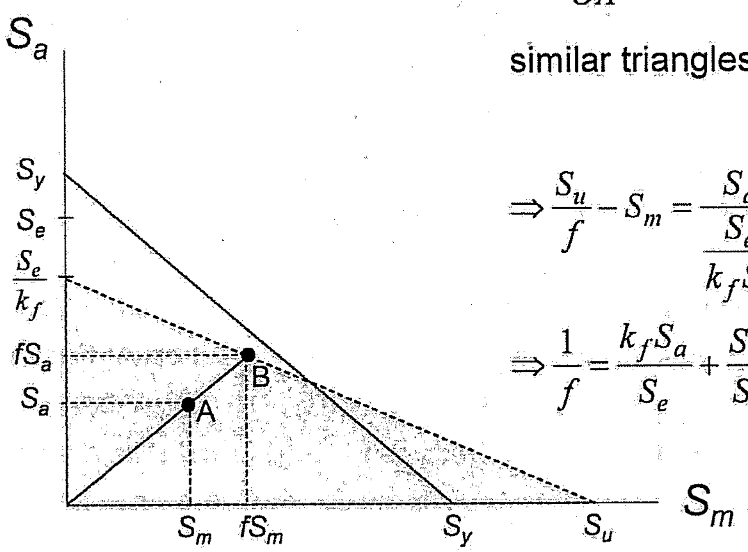
$$K_f = \frac{S_{a,smooth}}{S_{a,notch}}$$

However, this fatigue notch factor is also found to vary with both alternating and mean stress level and thus with number of cycles to failure.

**_S-N_ Design Procedure for Fatigue**

Constant life diagrams plotted as $S_a$ versus $S_m$, also called Goodman diagrams, can be used in design to give safe estimates of fatigue life and loads.

$$f = \frac{OB}{OA} = \text{factor of safety}$$




similar triangles $\rightarrow \dfrac{S_e}{k_f S_u} = \dfrac{fS_a}{S_u - fS_m} = \dfrac{S_a}{\frac{S_u}{f} - S_m}$

$$\Rightarrow \frac{S_u}{f} - S_m = \frac{S_a}{\frac{S_e}{k_f S_u}}$$

$$\Rightarrow \frac{1}{f} = \frac{k_f S_a}{S_e} + \frac{S_m}{S_u}$$

[16 marks]

### Linear elastic fracture mechanics (LEFM)

i) The critical stress, which causes a crack to propagate in an unstable fashion, giving fracture, is governed by the following relationships

$$\sigma\sqrt{\pi a} = \sqrt{EG_c} \quad \text{(plane stress)}$$

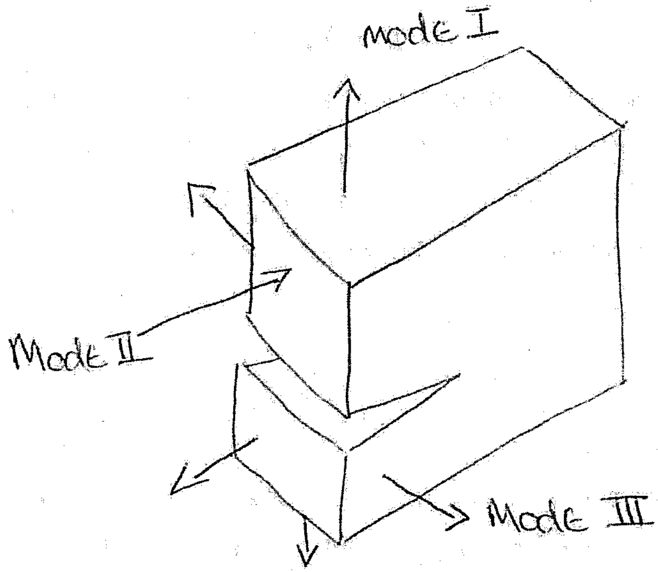
$$\sigma\sqrt{\pi a} = \sqrt{\frac{EG_c}{1-\nu^2}} \quad \text{(plane strain)}$$

Since the term on the right-hand side of these equations is a material constant and since the term on the left hand side is so common, it is usually abbreviated to the symbol, $K$, which is referred to as the *stress intensity factor* and the equations can be re-expressed as:

$$K = K_c$$

where $K_c$ is called the *critical stress intensity factor* or the *fracture toughness*. Thus $K_c = \sqrt{EG_c}$ .

ii) There are three different loading modes considered in fracture mechanics.

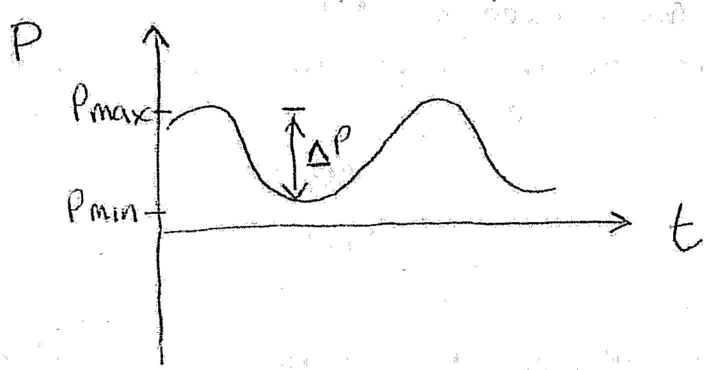



Mode I – opening mode
Mode II – shearing mode
Mode III – tearing mode

**iii) Fatigue crack growth:** It has been shown by Paris and co-workers (1961) that, for a wide range of conditions, there is a logarithmic linear relationship between crack growth rate and the stress intensity factor range during cyclic loading of cracked components



9




Variation of P(load) with t(time)

Considering a load cycle as shown above which gives rise to a load range

$$\Delta P = P_{\max} - P_{\min}$$

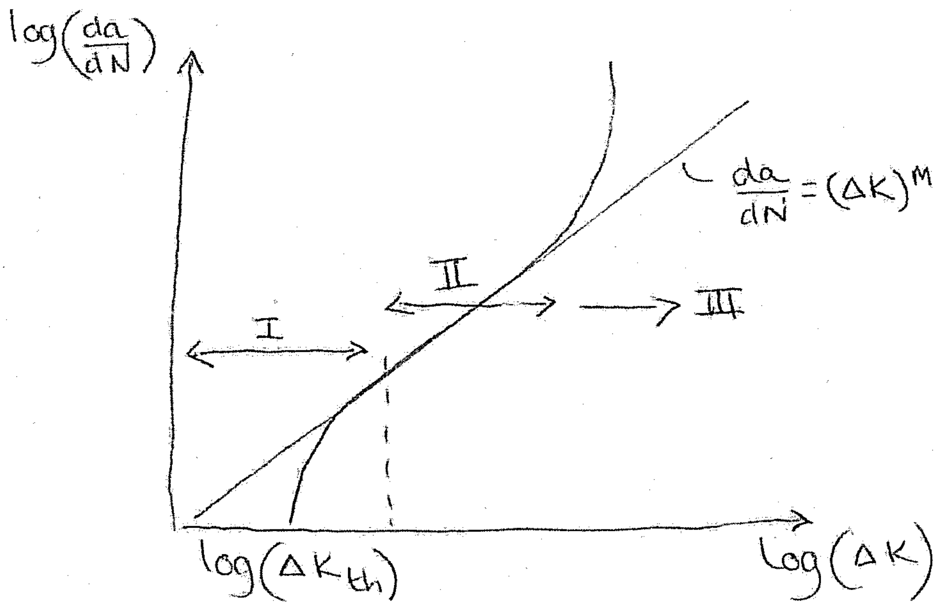
acting on a cracked body. The load range and crack geometry gives rises to a cyclic variation in stress intensity factor, which is given by

$$\Delta K = K_{\max} - K_{\min}$$

Even though the stress intensity factor may be less than the critical stress intensity factor for unstable crack growth, stable crack growth may occur if the stress intensity range, $\Delta K$, is greater than an empirically-determined material property called the *threshold stress intensity factor range*, designated $\Delta K_{th}$. In addition, Paris showed that the subsequent crack growth can be represented by an empirical relationship as follows:

$$\frac{da}{dN} = C\,(\Delta K)^m$$

where $C$ and $m$ are empirically-determined material constants. This relationship is known as the Paris equation. Fatigue crack growth data is often plotted as the logarithm of crack growth per load cycle, d$a$/d$N$, and the logarithm of stress intensity factor range. There are three stages. Below $\Delta K_{th}$., no observable crack growth occurs; region II shows an essentially linear relationship between log(d$a$/d$N$) and log($\Delta K$), where $m$ is the slope of the curve and $C$ is the vertical axis intercept; in region III, rapid crack growth occurs and little life is involved. Region III is primarily controlled by $K_c$ or $K_{Ic}$.




Typical (schematic) variation of log (da/dN) with log (ΔK)

The linear regime (Region II) is the region in which engineering components which fail by fatigue propagation occupy most of their life.

[17 marks]



4